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Two-Dimensional Radiative Transfer in a Cylindrical Layered Medium with Reflecting Interfaces

N. M. Reguigui* and R. L. Dougherty†
Oklahoma State University, Stillwater, Oklahoma 74078

A system of exact linear integral equations for the source function, intensity, and flux is presented for a two-dimensional cylindrical medium consisting of up to four layers with reflecting interfaces between the layers. Properties that may change from layer to layer are the single scattering albedo, optical thickness, and refractive index. The incident radiation is collimated and has a Bessel function distribution. The Bessel function boundary condition reduces the two-dimensional problem to a one-dimensional problem. Superposition is then used to derive the solution for any other boundary condition that is Hankel transformable. A special case of a Gaussian distribution that models a laser beam is presented. Some one-dimensional numerical results are presented for the source function and intensity within one- and two-layer media. Two-dimensional results are presented for back-scattered intensity due to the laser-beam boundary condition. Only the conservative case, optical thicknesses of 2.0 and 5.0, and refractive indices of 1.00 and 1.33 are considered.

Nomenclature

A_i = multiple reflection coefficient, defined in the Appendix
B_{ij} = function defined in the Appendix
D_N = function defined in the Appendix
E = component of kernel function for source function integral equation
G_{ij} = function defined in the Appendix
g = Hankel transformed function
I_i = intensity of radiation in layer i
I_i^+ = intensity in layer i in $+\tau_z$ direction
I_i^- = intensity in layer i in $-\tau_z$ direction
I_o = magnitude of incident intensity outside the medium
I_T = intensity transmitted across an interface
I_t = magnitude of incident intensity
J_0 = zeroth order Bessel function of first kind
K_{ij} = function defined in the Appendix
N = total number of layers
n_{ij} = ratio of layer i refractive index to that of layer j
P = scattering phase function
Q_{ij} = function defined in the Appendix
q_{zi} = z direction flux in layer i
R_i = function defined in the Appendix
r = radial distance from center of medium
r_o = laser beam radius
S_i = source function in layer i
S_{io}^c = source function lead term, defined in the Appendix
T = transmission function, defined by Eq. (32)
T_i = function defined in the Appendix
t_{ij} = transmission through the interface between layer i and j
x = dummy integration variable
Z_i = function defined in the Appendix
z = physical depth coordinate
β = Hankel transform parameter
β_e = extinction coefficient
$\Delta\tau_i$ = optical thickness of layer i, $\tau_{zi} - \tau_{zi-1}$
δ = Dirac delta function
δ_{ij} = Kronecker delta function
Θ = angle between the incoming and scattered radiation
θ = polar angle
Λ_{ij} = kernel function
μ_{ext} = cosine of the polar angle outside the medium
μ_i = cosine of the polar angle, θ, within layer i
$\mu_{ij,\text{cr}}$ = cosine of critical polar angle between layers i and j
μ_o = cosine of incident polar angle of intensity
ξ_{ij} = function defined in the Appendix
ρ_{ij} = Fresnel's coefficient of reflectivity for energy crossing the interface between layer i and layer j
τ_o = overall optical thickness of all layers of a given medium, $\tau_{zN} - \tau_{z0}$
τ_r = optical radial coordinate, $\beta_e r$
τ_{ro} = laser beam optical radius, $\beta_e r_o$
τ_s = optical coordinate along the general three-dimensional s direction
τ_z = optical depth coordinate, $\beta_e z$
τ_{zi} = optical depth at the bottom of layer i
τ_{zi-1} = optical depth at the top of layer i
Φ_{ij} = function defined in the Appendix
ϕ = azimuthal angle
ψ_i = function defined in the Appendix
Ω = solid angle
ω_i = single scattering albedo in layer i

Subscripts

cr = related to critical angle
ext = external to all layers of the medium
i = layer i
j = layer j
r = related to radial direction
s = related to a general three-dimensional direction
z = related to depth direction
β = Hankel transformed variable

Superscripts

1,2 = either function 1 or 2 (not both at once)
+, − = either positive (+) or negative (−) z direction (not both at once)

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*Graduate Student, School of Mechanical and Aerospace Engineering, EN 218.
†Associate Professor, School of Mechanical and Aerospace Engineering, EN 218. Member AIAA.

Introduction

IN radiative transfer problems, many realistic situations, such as solar pond modeling, laser-material interaction, and underwater radiation back scattering, require a multidimensional analysis that accounts for the effects of the refractive index. However, most of the results found in the literature employ many simplifying assumptions, such as one-dimensional geometries or multidimensional geometries that neglect the effects of the index of refraction and consider only one or two aspects of the radiative transfer problem of interest: reflecting boundaries,[1-4] diffusely incident radiation[1] and collimated incident radiation,[4] isotropic scattering[5] and anisotropic scattering,[6] variation in absorption and scattering with depth,[7,8] and two or three dimensionality.[5,6,8] A more realistic analysis has to include the effects of multidimensionality as well as the effects of possible variations in the pertinent physical properties, such as index of refraction, and absorption and scattering coefficients of the medium.

The index of refraction change across an interface, coupled with the directional redistribution of radiation by anisotropic scattering, has been shown to alter considerably the radiative field within the medium.[1] Cengel and Ozisik[2] presented one-dimensional data for a one-layer slab which indicate that the constant reflectivity approximation often leads to severalfold under- or overestimated results as compared to the exact handling of the reflection coefficient; thus disproving the assumption of using average values.

Most of the work that includes the effect of index of refraction has been limited to one dimension. Buckius and Tseng[1] considered a plane-parallel isothermally emitting medium that scatters anisotropically. Armaly and Lam[3] and Dougherty[4] have investigated the influence of refractive index on the reflectance from a semi-infinite one-dimensional absorbing-scattering medium.

Although a lot of work has been done for the two-dimensional geometry, index of refraction effects have been neglected for much of that work. A two-dimensional problem that has received attention in the past 10 years is the back scattering of a laser beam by a multiple scattering medium when the incident beam is normal to the surface of the medium.[5,6] Most of these studies were based on the assumption that the index of refraction is unity. Crosbie and Dougherty presented graphical and tabular results for isotropic[5] and anisotropic[6] scattering in a finite cylindrical medium exposed to a laser beam. This present work builds on these studies to include the effects of the index of refraction and the effects of layering the medium.

Some investigators[9,10] have considered superposing the unit refractive index results or introducing a factor in the results based on the unit index of refraction to account for index of refraction effects. Crosbie and Dougherty[9] have proposed superposing the unit refractive index results to obtain directional and hemispherical reflectances from a two-dimensional cylindrical medium. No numerical results were presented, but some asymptotic solutions were examined. In a comparison between theoretical and experimental results for back scattering from a two-dimensional optically thick medium exposed to a laser beam, Nelson et al.[10] have found that agreement between theory and experiment is improved when the theoretical back-scattered intensity for unit refractive index is reduced by $(1 - \rho_N)^2/n^2$. However, this approximation was found to be valid only in the thin limit.

Studies of radiative transfer in composite (layered) media including the effects of scattering are very limited and mostly deal with one-dimensional situations. Shouman and Ozisik[11] considered the problem of radiative transfer in an absorbing, emitting, and isotropically scattering two-layer slab with diffusely and specularly reflecting boundaries. The slab was irradiated externally by diffuse radiation at the boundary surface. The F_N method was used to compute results for the transmissivity and reflectivity of the slab. The results showed that by increasing the single scattering albedo of the second slab, the reflectivity and the transmissivity of the composite slab increase. The transmissivity was found also to be influenced by the relative optical thicknesses of the slabs.

A few other studies also exist for media with layered properties; but usually the index of refraction is assumed to be unity, which considerably simplifies the analysis. Stamnes and Conklin[7] have developed a matrix method to solve the discrete ordinate approximation to the radiative equation in a homogeneous, plane parallel atmosphere consisting of N adjacent layers. However, because of the one-dimensionality of the analysis and the unit index of refraction, this study is applicable to the field of atmospheric physics and meteorology.

A general multilayer study was done by Sutton and Kamath.[8] They considered a three-dimensional rectangular medium with layered properties and unit index of refraction. The boundaries between the layers were considered transparent, hence neglecting the effects of index of refraction changes. The volume-averaged discrete ordinates method was employed to solve for the transmitted and reflected radiative flux.

In the present investigation, the exact solution for the radiative transport equation within an absorbing and isotropically scattering two-dimensional cylindrical medium is presented. The medium is finite in height and infinite in the radial direction, has internally reflecting top and bottom surfaces, and consists of layers whose single scattering albedo, optical thickness, and index of refraction may differ (see Fig. 1). The interfaces are assumed to be smooth so that Fresnel's equation and Snell's law can be used. The incident radiation is collimated and normal to the top surface of the medium and it is assumed to be azimuthally symmetric. It has a radial variation in the form of a Bessel function. The Bessel function boundary condition is used to separate variables, and therefore reduces the two-dimensional equations to one-dimensional form. Superposition is then used to solve the problem in which the incident radiation is in the form of a Gaussian function (i.e., a laser beam). No radiation is incident on the bottom surface of the medium.

Development

The transfer equation in a given coordinate system for a certain layer i in a medium consisting of many layers and in which scattering and absorption take place is[5]

$$\frac{dI_i(\tau_s)}{d\tau_s} + I_i(\tau_s) = S_i(\tau_s) = \frac{\omega_i}{4\pi}\int_{4\pi} I_i(\tau_s)P(\cos\Theta)\, d\Omega \qquad (1)$$

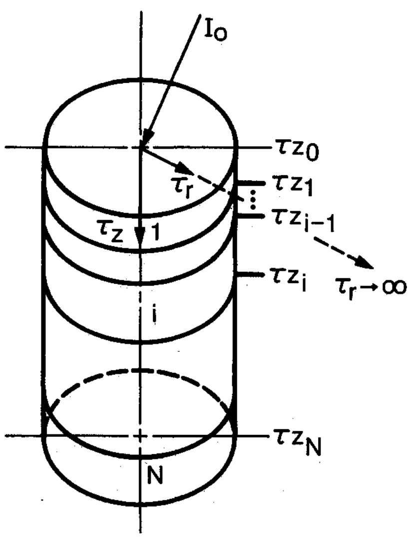

Fig. 1 Geometry of a layered medium, infinite in the τ_r direction and finite in the τ_z direction.

In the source function S_i, ω_i is assumed to be constant for layer i, but could differ among the layers. For the remainder of the analysis, P will be assumed equal to one for isotropic scattering.

Decomposing Eq. (1) into its positive (I^+) and negative (I^-) directions, and then solving both equations using an integrating factor approach, yields[5]

$$I_i^+(\tau_r,\tau_z,\mu_i,\phi) = I_i^+(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi) \times \exp[(\tau_{zi-1} - \tau_z)/\mu_i] + \int_{\tau_{zi-1}}^{\tau_z} \exp[(\tau'_z - \tau_z)/\mu_i] \times S_i(\tau'_r,\tau'_z)\, d\tau'_z/\mu_i \tag{2}$$

and

$$I_i^-(\tau_r,\tau_z,\mu_i,\phi) = I_i^-(\tau'_{ri},\tau_{zi},\mu_i,\phi)\exp[-(\tau_{zi} - \tau_z)/\mu_i] + \int_{\tau_z}^{\tau_{zi}} \exp[-(\tau'_z - \tau_z)/\mu_i]S_i(\tau'_r,\tau'_z)\, d\tau'_z/\mu_i \tag{3}$$

where τ'_r is given by

$$(\tau'_r)^2 = \tau_r^2 + a^2 - 2a\tau_r\cos(\phi') \tag{4}$$

with

$$a^2 = \tau_\alpha^2 = (\tau'_z - \tau_z)^2(1 - \mu_i'^2)/\mu_i'^2 \tag{5}$$

and $\tau'_{ri} = \tau'_r$ for $\tau_z = \tau_{zi}$ (see Fig. 2). All of the quantities with subscript i are related to layer i, and τ_{zi} and τ_{zi-1} refer to the bottom and top of the layer i, respectively (see Fig. 1). S is a function of τ_r and τ_z in Eqs. (2) and (3) only because of the isotropy of the problem and the axisymmetric loading assumption. $I_i^+(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi)$ and $I_i^-(\tau'_{ri},\tau_{zi},\mu_i,\phi)$ are the boundary conditions at the top and bottom of layer i, repectively, which is finite in the depth (τ_z) direction and infinite in the radial (τ_r) direction. Substituting Eqs. (2) and (3) into the right-hand side of Eq. (1) and rearranging the result gives the integral equation for the source function in layer i

$$S_i(\tau_r,\tau_z) = \frac{\omega_i}{4\pi}\int_0^{2\pi}\int_0^1 \{I_i^+(\tau'_{ri-1},\tau_{zi-1},\mu',\phi') \times \exp[(\tau_{zi-1} - \tau_z)/\mu'_i] + I_i^-(\tau'_{ri},\tau_{zi},\mu',\phi') \times \exp[-(\tau_{zi} - \tau_z)/\mu'_i]\}\, d\mu'_i\, d\phi' + \frac{\omega_i}{4\pi}\int_0^{2\pi}\int_0^1\int_{\tau_{zi-1}}^{\tau_{zi}} \times \exp[-|\tau'_z - \tau_z|/\mu'_i]S_i(\tau'_r,\tau'_z)\, d\tau'_z\, d\mu'_i\, d\phi'/\mu'_i \tag{6}$$

Equation (6) actually represents a set of equations to be solved for all layers in the medium. Besides the top and bottom boundary conditions of the medium which should be known,

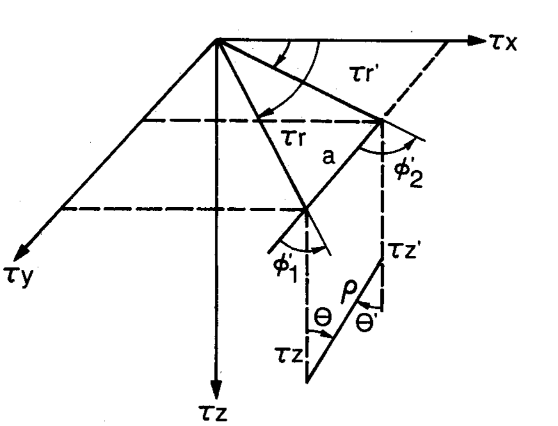

Fig. 2 Cylindrical coordinate system.

the intensities at the interior interfaces between the different layers are unknown, and they need to be eliminated from the set of Eq. (6) in order to solve for the source functions for all layers.

Intensities and Source Functions

We will consider collimated incident radiation at the topmost boundary. Thus, the radiation crossing the interface of an arbitrary layer i has, in general, a collimated part, due to the collimated incident radiation, and a diffuse part due to scattering from the adjacent layers. The collimated radiation can be represented using the delta function, i.e.

$$I_i(\mu'_i,\phi') = I_i\delta(\mu'_i - \mu_i)\delta(\phi' - \phi) \tag{7}$$

I_i is the magnitude of the intensity and (μ_i,ϕ) determines the direction of the collimated radiation. From the law of conservation of energy, the relationship between the incident and the refracted intensities is[3]

$$I_j = \mu_i[1 - n_{ij}^2(1 - \mu_i^2)]^{-1/2}[1 - \rho_{ij}(\mu_i)]I_i \tag{8}$$

where I_j is the magnitude of the intensity refracted into layer j. The reflectivity function ρ_{ij} is given by Fresnel's equation[1] and μ_j is given by (Snell's Law[1])

$$\mu_j = [1 - (1 - \mu_i^2)n_{ij}^2]^{1/2} \tag{9}$$

The imaginary part of the refractive index is assumed to be small relative to the real part, which is true for most dielectrics. When $n_{ij} > 1$, Eq. (9) cannot be satisfied for $\mu_i < \mu_{ij,\mathrm{cr}}$ where

$$\mu_{ij,\mathrm{cr}}^2 = 1 - n_{ij}^2 \tag{10}$$

Therefore, when $n_{ij} > 1$, all intensity that is coming at angles θ_i greater than $\theta_{ij,\mathrm{cr}}$ (or $\mu_i < \mu_{ij,\mathrm{cr}}$) will be totally reflected ($\rho_{ij} = 1.0$).

When the intensity crossing the interface between layers i and j is diffuse, the amount of the refracted intensity in medium j is[3]

$$I_j(\mu_j) = n_{ji}^2[1 - \rho_{ij}(\mu_i)]I_i(\mu_i) \tag{11}$$

In Eq. (2) $I_i^+(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi)$ is the sum of the incident radiation that crosses the interface and the radiation originating in the layer that is reflected at the top interface in the positive direction. Therefore

$$I_i^+(\tau'_{ri-1},\tau'_{zi-1},\mu_i,\phi) = I_T^+(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi) + \rho_{i,i-1}(\mu_i)I_i^-(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi) \tag{12}$$

Similarly, the boundary condition $I_i^-(\tau'_{ri},\tau_{zi},\mu_i,\phi)$ in Eq. (3) is the sum of the interface transmittance of radiation from layer $i + 1$ plus the interface back reflection of scattered radiation at the bottom of layer i. Therefore

$$I_i^-(\tau'_{ri},\tau_{zi},\mu_i,\phi) = I_T^-(\tau'_{ri},\tau_{zi},\mu_i,\phi) + \rho_{i,i+1}(\mu_i)I_i^+(\tau'_{ri},\tau_{zi},\mu_i,\phi) \tag{13}$$

In Eq. (12), $I_i^-(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi)$ is replaced by Eq. (3), where τ_z is set equal to τ_{zi-1}. The transmitted intensity $I_T^+(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi)$ could be collimated and/or diffuse. Similarly, in Eq. (13), $I_i^+(\tau'_{ri},\tau_{zi},\mu_i,\phi)$ is written from Eq. (2) where τ_z is set equal to τ_{zi} and $I_T^-(\tau'_{ri},\tau_{zi},\mu_i,\phi)$ is determined from Eq. (8) and/or Eq. (11). Both resulting equations can be solved simultaneously for $I_i^+(\tau'_{ri-1},\tau_{zi-1},\mu_i,\phi)$ and $I_i^-(\tau'_{ri},\tau_{zi},\mu_i,\phi)$ [using Eqs. (2) and (3) a second time in the process]. This will result in a system of linear simultaneous equations in terms of the interface intensities. When the solution to this sytem is put back into Eqs. (2) and (3), we get

$$I_i^+(\tau_r,\tau_z,\mu_i) = \frac{A_i(\mu_i)}{D_N(\mu_i)} \exp[(\tau_{zi-1} - \tau_z)/\mu_i]$$
$$\times \Bigg[Z_i^+(\mu_i) I_o(\tau'_{ri-1}) \delta(\mu_o - 1)$$
$$+ \Bigg(R_i^2(\mu_i) \exp(-\Delta\tau_i/\mu_i) \sum_{j=i+1}^{N} \boldsymbol{B}_{ij}^1(\mu_i,\tau'_r)$$
$$+ T_i^1(\mu_i) \sum_{j=1}^{i-1} \boldsymbol{B}_{ij}^2(\mu_i,\tau'_r) \Bigg) \Bigg]$$
$$+ \frac{A_i(\mu_i)}{D_N(\mu_i)} \exp[(\tau_{zi-1} - \tau_z)/\mu_i] R_i^2(\mu_i)$$
$$\times \int_{\tau_{zi-1}}^{\tau_{zi}} S_i(\tau'_r,\tau'_z) \psi_i^1(\mu_i,\tau'_z)\, d\tau'_z/\mu_i$$
$$+ \int_{\tau_{zi-1}}^{\tau_z} \exp[(\tau'_z - \tau_z)/\mu_i] S_i(\tau'_r,\tau'_z)\, d\tau'_z/\mu_i \quad (14)$$

and

$$I_i^-(\tau_r,\tau_z,\mu_i) = \frac{A_i(\mu_i)}{D_N(\mu_i)} \exp[-(\tau_{zi} - \tau_z)/\mu_i]$$
$$\times \Bigg[Z_i^-(\mu_i) I_o(\tau'_{ri}) \delta(\mu_o - 1) + \Bigg(R_i^1(\mu_i)$$
$$\times \exp(-\Delta\tau_i/\mu_i) \sum_{j=1}^{i-1} \boldsymbol{B}_{ij}^2(\mu_i,\tau'_r)$$
$$+ T_i^2(\mu_i) \sum_{j=i+1}^{N} \boldsymbol{B}_{ij}^1(\mu_i,\tau'_r) \Bigg) \Bigg]$$
$$+ \frac{A_i(\mu_i)}{D_N(\mu_i)} \exp[(\tau_{zi} - \tau_z)/\mu_i] R_i^1(\mu_i)$$
$$\times \int_{\tau_{zi-1}}^{\tau_{zi}} S_i(\tau'_r,\tau'_z) \psi_i^2(\mu_i,\tau'_z)\, d\tau'_z/\mu_i$$
$$+ \int_{\tau_z}^{\tau_{zi}} \exp[-(\tau'_z - \tau_z)/\mu_i] S_i(\tau'_r,\tau'_z)\, d\tau'_z/\mu_i \quad (15)$$

where because of the axisymmetric loading, the intensity does not depend on ϕ directly, except through τ_r. The equations for $A_i(\mu_i)$, $D_N(\mu_i)$, $Z_i^+(\mu_i)$, $Z_i^-(\mu_i)$, $R_i^1(\mu_i)$, $R_i^2(\mu_i)$, $T_i^1(\mu_i)$, $T_i^2(\mu_i)$, $\boldsymbol{B}_{ij}^1(\mu_i,\tau_r)$, $\boldsymbol{B}_{ij}^2(\mu_i,\tau_r)$, $\psi_i^1(\mu_i,\tau_z)$, and $\psi_i^2(\mu_i,\tau_z)$ are given in the Appendix. The first lead term in each of Eqs. (14) and (15) has an infinite value when $\mu_o = 1$ due to the Dirac delta. The remaining terms in the equations include the radiation contributions from all the layers other than layer i, as well as the contribution from within the layer. Notice that when the number of layers is greater than one, τ'_r will be given by a more general formula than Eq. (4), i.e., Eq. (A28).

In the process of obtaining Eqs. (14) and (15), the following equation for the source function was obtained:

$$S_i(\tau_r,\tau_z) = \frac{\omega_i}{4\pi} S_{io}^c(1,\tau_z) I_o(\tau_r) + \frac{\omega_i}{4\pi} \sum_{j=1}^{N} \int_{\tau_{zj-1}}^{\tau_{zj}} \int_0^{2\pi}$$
$$\times \int_{\mu_{ij,cr}}^{1} S_j(\tau'_r,\tau'_z)\, d\tau'_z\, d\phi'\, d\mu'_j/\mu'_j$$
$$\times \left\{ \frac{A_i(\mu'_i)}{D_N(\mu'_i)} \Gamma_{ij}(\mu'_i,\tau_z,\tau'_z) + \delta_{ij} \exp[-|\tau'_z - \tau_z|/\mu'_i] \right\} \quad (16)$$

Equation (16) represents a system of simultaneous integral equations in terms of the source functions of each layer for a cylindrical medium consisting of N layers. $S_{io}^c(\mu_i,\tau_z)$ is an initial term due only to the collimated boundary condition, and it is being evaluated at $\mu_i = 1$ because of the normal incidence of the radiation [see Eq. (A1)]. Again, μ_i should be greater than the critical angle $\mu_{ij,cr}$ when considering radiation from layers other than layer i. The auxiliary function Γ_{ij} is another collection of terms due to the transmission, reflection, and attenuation of the radiation as it crosses the different layers, and it depends on the relation between i and j as given in the Appendix [Eqs. (A16)–(A18)]. Note that Eqs. (14), (15), and (16) reduce to those of Buckius and Tseng[1] when there is only one layer (except that their boundary condition is diffuse while this boundary condition is collimated).

Radiative Flux

Once the intensity of radiation is determined, the radiative flux in the z direction may be computed from substituting Eqs. (14) and (15) into the following equation:

$$q_{zi}(\tau_r,\tau_z) = \int_0^{2\pi} \int_0^1 I_i^+(\tau'_r,\tau_z,\mu',\phi')\mu'_i\, d\mu'_i\, d\phi'$$
$$- \int_0^{2\pi} \int_0^1 I_i^-(\tau'_r,\tau_z,\mu',\phi')\mu'_i\, d\mu'_i\, d\phi' \quad (17)$$

Having the basic two-dimensional equations derived for the collimated boundary condition, it is easy to write those equations for other forms of boundary conditions. This is explained in more detail in the next section.

Special Boundary Conditions

Bessel Function Boundary Condition

Even for the isotropic case, the two-dimensional equations are still difficult to solve. However, if the incident radiation is assumed to be in the form of a Bessel function, then the source function, the intensity, and the flux can be shown[5] to be separable functions of τ_r and τ_z, and the two-dimensional governing equations can be reduced to simpler one-dimensional forms. The Bessel solution can then be superposed to obtain the solution for any Hankel transformable boundary condition.[5]

A Bessel function boundary condition has the following form:

$$I_o(\tau_r) = I_t J_0(\beta\tau_r) \quad (18)$$

where I_t is a constant, the magnitude of the intensity.

Source Function

The Bessel-varying boundary condition suggests that, for this case, the source function can be separated,[5] i.e.

$$S_i(\tau_r,\tau_z) = (I_t\omega_i/4\pi) J_0(\beta\tau_r) S_{\beta,i}(\tau_z) \quad (19)$$

and $S_{\beta,i}(\tau_z)$ represents the β-varying one-dimensional source function for layer i. Substituting Eqs. (18) and (19) into Eq. (16) and then, using a technique similar to Crosbie and Dougherty,[6] the β-varying source function can be found[12,13]

$$S_{\beta,i}(\tau_z) = S_{io}^c(1,\tau_z) + \frac{1}{2} \sum_{j=1}^{N} \omega_j \int_{\tau_{zj-1}}^{\tau_{zj}}$$
$$\times S_{\beta,j}(\tau'_z) \Lambda_{ij}(\beta,\tau_z,\tau'_z)\, d\tau'_z \quad (20)$$

where

$$\Lambda_{ij}(\beta,\tau_z,\tau_z') = \delta_{ij}E(\beta,\tau_z - \tau_z') + \int_{\mu_{ij,cr}}^{1} J_o(\beta\tau_{\alpha ij}) \times \frac{A_i(\mu_i')}{D_N(\mu_i')}\Gamma_{ij}(\mu_i',\tau_z,\tau_z')\,d\mu_i'/\mu_j' \quad (21)$$

where $\tau_{\alpha ij}$ is defined by Eq. (A29), and where it was necessary to split the integral over μ_i', because when j is different from i, μ_j' will be restricted to the interval between $\mu_{ij,cr}$ [given by Eq. (10)] and 1.0. E in Eq. (21) is a Bessel-varying exponential integral, and it is defined as

$$E(\beta,\tau) = \int_0^1 J_o\{\beta|\tau|[1 - \mu^2]^{1/2}/\mu\}\exp[-|\tau|/\mu]\,d\mu/\mu \quad (22)$$

which is related to the generalized exponential integral of Ref. 5. This integral reduces to the exponential integral when β is zero (the one-dimensional case). Γ_{ij} can be numerically computed for any given values of μ_i', τ_z, and τ_z' from Eqs. (A16)–(A18). Then Eq. (21) can be integrated numerically to compute Λ_{ij}, and this can be used in Eq. (20) to solve for the β-varying source function.

Intensity Equation

The Bessel-varying boundary condition suggests that the intensity can be separated,[5] i.e.

$$I_i(\tau_r,\tau_z,\mu,\phi) = I_rJ_o(\beta\tau_r)I_{\beta,i}(\tau_z,\mu,\phi) \quad (23)$$

where $I_{\beta,i}(\tau_z,\mu,\phi)$ represents the β-varying one-dimensional intensity for layer i. Similarly, the effective intensity terms B^1 and B^2 [of Eqs. (14) and (15), defined in Eq. (A19)] can be separated as

$$B_{ij}^{1,2}(\mu_i,\tau_r) = (I_r\omega_j/4\pi)J_o(\beta\tau_r)B_{\beta,ij}^{1,2}(\mu_i) \quad (24)$$

where $B_{\beta,ij}^{1,2}(\mu_i)$ is now defined in terms of the integral of $S_{\beta,j}(\tau_z)$, as in Eq. (A20). Substituting Eqs. (18), (23), and (24) into Eqs. (14) and (15) gives the β-varying intensity equation, viz.

$$I_{\beta,i}^{+}(\tau_z,\mu_i) = \frac{A_i(\mu_i)}{D_N(\mu_i)}\exp[(\tau_{zi-1} - \tau_z)/\mu_i] \times \left[Z_i^{+}(\mu_i)\delta(\mu_o - 1) + \frac{1}{4\pi}\left(R_i^2(\mu_i) \times \exp(-\Delta\tau_i/\mu_i)\sum_{j=i+1}^{N}\omega_jB_{\beta,ij}^1(\mu_i) + T_i^1(\mu_i) \times \sum_{j=1}^{i-1}\omega_jB_{\beta,ij}^2(\mu_i)\right)\right] + \frac{\omega_i}{4\pi\mu_i}\frac{A_i(\mu_i)}{D_N(\mu_i)}\exp[(\tau_{zi-1} - \tau_z)/\mu_i]R_i^2(\mu_i)\int_{\tau_{zi-1}}^{\tau_{zi}} S_{\beta,i}(\tau_z')\psi_i^1(\mu_i,\tau_z')\,d\tau_z' + \frac{\omega_i}{4\pi\mu_i}\int_{\tau_{zi-1}}^{\tau_z}\exp[(\tau_z' - \tau_z)/\mu_i]S_{\beta,i}(\tau_z')\,d\tau_z' \quad (25)$$

and

$$I_{\beta,i}^{-}(\tau_z,\mu_i) = \frac{A_i(\mu_i)}{D_N(\mu_i)}\exp[-(\tau_{zi} - \tau_z)/\mu_i] \times \left[Z_i^{-}(\mu_i)\delta(\mu_o - 1) + \frac{1}{4\pi}\left(R_i^1(\mu_i)\exp(-\Delta\tau_i/\mu_i) \times \sum_{j=1}^{i-1}\omega_jB_{\beta,ij}^2(\mu_i) + T_i^2(\mu_i)\sum_{j=i+1}^{N}\omega_jB_{\beta,ij}^1(\mu_i)\right)\right] + \frac{\omega_i}{4\pi\mu_i}\frac{A_i(\mu_i)}{D_N(\mu_i)}\exp[-(\tau_{zi} - \tau_z)/\mu_i]R_i^1(\mu_i) \times \int_{\tau_{zi-1}}^{\tau_{zi}} S_{\beta,i}(\tau_z')\psi_i^2(\mu_i,\tau_z')\,d\tau_z' + \frac{\omega_i}{4\pi\mu_i} \times \int_{\tau_z}^{\tau_{zi}}\exp[-(\tau_z' - \tau_z)/\mu_i]S_{\beta,i}(\tau_z')\,d\tau_z' \quad (26)$$

Flux

Similarly, the flux can be written as a separable function of τ_r and τ_z, i.e.

$$q_{zi}(\tau_r,\tau_z) = I_rJ_o(\beta\tau_r)q_{z\beta,i}(\tau_z) \quad (27)$$

Substituting this equation and Eqs. (23), (25), and (26) into Eq. (17), and following a similar development as for the source function gives

$$q_{z\beta,i}(\tau_z) = \frac{A_i(1)}{D_N(1)}[\exp[(\tau_{zi-1} - \tau_z)]Z_i^{+}(1) - \exp[-(\tau_{zi} - \tau_z)]Z_i^{-}(1)] + \frac{1}{2}\int_{\mu_{ij,cr}}^{1} \times \frac{A_i(\mu_i')}{D_N(\mu_i')}\mu_i'\left\{\sum_{j=i+1}^{N}\omega_jB_{q\beta,ij}^1(\mu_i',\tau_z) \times \left[R_i^2(\mu_i')\exp[(\tau_{zi-1} - \tau_z - \Delta\tau_i)/\mu_i'] - T_i^2(\mu_i') \times \exp[-(\tau_{zi} - \tau_z)/\mu_i']\right] + \sum_{j=1}^{i-1}\omega_jB_{q\beta,ij}^2(\mu_i',\tau_z) \times (T_i^1(\mu_i')\exp[(\tau_{zi-1} - \tau_z)/\mu_i'] - R_i^1(\mu_i') \times \exp[(\tau_z - \tau_{zi} - \Delta\tau_i)/\mu_i'])\right\}d\mu_i' + \frac{\omega_i}{2}\int_0^1\int_{\tau_{zi-1}}^{\tau_{zi}} \times S_{\beta,i}(\tau_z')J_o(\beta\tau_{\alpha ii})\left\{\mathrm{sgn}(\tau_z - \tau_z') \times \exp[-|\tau_z' - \tau_z|/\mu_i'] + \frac{A_i(\mu_i')}{D_N(\mu_i')}\left[\exp[(\tau_{zi-1} - \tau_z)/\mu_i']\,R_i^2(\mu_i')\psi_i^1(\mu_i',\tau_z') - \exp[-(\tau_{zi} - \tau_z)/\mu_i']\,R_i^1(\mu_i')\psi_i^2(\mu_i',\tau_z')\right]\right\}d\tau_z'\,d\mu_i' \quad (28)$$

where $B_{q\beta}$ is a modification of the function B_β, given in Eq. (A20), replacing $S_{\beta,j}(\tau_z')$ by $S_{\beta,j}(\tau_z')J_o(\beta\tau_{\alpha ij})$.

Other Boundary Conditions

After the basic solution for the Bessel function boundary condition has been found, this can be used in superposition to find the solution for any other boundary condition that is expressible in terms of a Hankel transform.[5] Representation of the τ_r-varying boundary condition, $I_o(\tau_r)$, by a Hankel transform yields[5]

$$I_o(\tau_r) = \int_0^\infty \beta J_0(\beta\tau_r)\left[\int_0^\infty tJ_0(\beta t)I_o(t)\,dt\right]d\beta \quad (29)$$

The solution of any other problem with a boundary condition that is Hankel transformable can be found from an inversion process similar to Eq. (29)—substituting the source function or flux for intensity as required.

Gaussian Distribution

In this work, a Gaussian functional form is used to model a laser beam,[5] such that $I_o(\tau_r)$ and is

$$I_o(\tau_r) = I_t \exp(-\tau_r^2/\tau_{ro}^2) \quad (30)$$

where τ_{ro} is the effective optical radius of the beam (at the e^{-1} point). The source function inversion equation becomes[12]

$$S_i(\tau_r,\tau_z) = (\omega_i I_t/8\pi)(r_o/r)^2 \int_0^\infty xJ_0(x) \times \exp[-(xr_o/2r)^2] S_{x/\tau_r,i}(\tau_z)\, dx \quad (31)$$

where the integration over β has been replaced by integration over x ($=\beta_{\tau_r}$), and τ_{ro}/τ_r has been replaced by r_o/r (assuming uniform properties within each layer). The corresponding intensity and flux equations are similar to Eq. (31).[12,13]

Results

Numerical Procedure

The method of solution consists basically of using Gaussian quadrature to do the different integrals involved, and applying successive approximation to determine the source function of Eq. (20). First, the different $\Lambda_{ij}(\beta,\tau_z,\tau_z')$ values given by Eq. (21) are computed at the quadrature points τ_z and τ_z' for every β. These values necessitate a large amount of storage, about one-third megabyte per β for a one-layer problem. In addition to storage, the computations require approximately 45 CPU minutes on an IBM 3090 per β calculation for a one-layer problem. Increasing the number of layers obviously increases this computational time. But once the Λ_{ij} values are computed, the iterations to compute S_β become quite fast. Also, since the Λ_{ij} values do not depend on albedo (ω), the source function can be determined for a variety of ω values without having to evaluate the Λ_{ij} values for every ω. To start the iteration on Eq. (20), S is first set equal to $S_{io}^c(1,\tau_z)$ of Eq. (A1).

Test runs have shown that the function $\Gamma_{ij}(\mu,\tau_z,\tau_z')$ has several irregularities in the curve as a function of the cosine of the angle μ. This is due to the sharp change in the reflectivity coefficient function[1] whenever the argument (a function of μ) becomes larger than the appropriate critical angle. To alleviate this problem, the integral was divided into several subintervals determined by all of the critical angles that can exist when going from layer i to layer j; and a dense quadrature was placed around each of these breakpoints. This procedure increased the execution time considerably, but on the other hand, helped to assure the accuracy of the results. Convergence in computing the source function was achievable for any accuracy desired and for any set of variables ($\omega,n,\Delta\tau,\ldots$). However, for albedo equal to unity, the convergence was relatively slower than for other values.

To test the accuracy of the computer program, some published results were first reproduced. Then, some particular two-dimensional multilayer problems were considered. Most of the cases from the literature that have been considered are in good agreement with the generated results. Unfortunately, all of those cases deal with one-dimensional geometries, as there is no other work in the open literature that considers index of refraction effects for two-dimensional geometries.

One-Dimensional Results

Two main problems were considered for comparison. The first one considers a single-layer one-dimensional medium with a top reflecting surface.[4] The second case has, in addition to the first case, a bottom reflecting surface.[14] Both cases have collimated intensity incident on the top surface and no incident intensity on the bottom surface of the medium. By setting β equal to zero and the number of layer (N) to one, the current results were comparable to one-dimensional single-layer results. Tables 1 and 2 present a comparison of results generated by this multilayer program with those of Dougherty.[4] In those tables, "Direct" and "Ambar" refer to solution methods employed by Dougherty, solving the integral equation for the source function by discretizing τ_z (Direct), and solving only for the source function at the boundary by a variation of Ambartzumian's method (Ambar). Table 3 gives a comparison between the current results and those of Dougherty[14] for a one-layer medium with both boundaries having reflective interfaces. Current 1L and Current 2L refer to employing the modeling of this paper, and using either one uniform layer (1L) of a given number of τ_z quadrature or two uniform layers (2L) having the same properties, but twice the number of τ_z quadrature. In addition, Table 3 uses $T(\mu)$, which is given by[12]

$$T(\mu) = [4\pi\mu/t_{01}(1)][I^+_{\beta=0,1}(\tau_o,\mu) - \rho_{10}(\mu)\exp(-\tau_o/\mu)I^-_{\beta=0,1}(0,\mu)] \quad (32)$$

In all of these tables, the maximum error between any two entries is less than 0.07%. If the layer is divided into two sublayers that have the same properties, then the two-layer model (Current 2L) predicts the same values as compared to those from the one-layer model (Current 1L) as shown in the tables. The slight differences in results between the 1L cases and the 2L cases are due to the increased number of quadrature for τ_z.

Some additional one-dimensional results will be presented first in order to show the effects of changing the index of refraction between two layers. The one-dimensional results are a special case ($\beta = 0$) for the β-varying solution. Figure 3 presents a plot of the source function as a function of the medium depth (τ_z). The medium consists of two layers. The top layer models a thin film of paraffin oil[15] ($n_1 = 1.48$) with an optical thickness of 0.1. The second layer has an optical thickness of 5.0 and an index of refraction of 1.33 (water). The albedo is assumed to be unity for the water and 0.9 for the oil. These albedos can be experimentally simulated by adding small latex spheres to the fluids.[15] The graph shows a

Table 1 Comparison of current results with those of Dougherty[4]

	$S_{\beta=0,1}(0)/t_{01}(1)$	$40\pi\mu I^-_{\beta=0,1}(0,\mu)/t_{01}(1)$, $\mu = 0.5$	$40\pi\mu I^-_{\beta=0,1}(0,\mu)/t_{01}(1)$, $\mu = 1.0$	$10[1.0 - q_{z\beta=0,1}(0)]$
Direct[4]	1.486318	2.72818	4.25353	0.60330
Ambar[4]	1.486321	2.7282	4.25366	0.60331
Current 1L	1.485868	2.72752	4.25246	0.60315
Current 2L	1.486116	2.72772	4.25710	0.60323

$n_{10} = 1.33$, $n_{12} = 1.00$, $\tau_o = 6.0$, $\omega_1 = 0.5$, $\beta = 0.0$.

Table 2 Comparison of current results with those of Dougherty[4]

	$S_{\beta=0,1}(0)/t_{01}(1)$	$40\pi\mu I^-_{\beta=0,1}(0,\mu)/t_{01}(1)$, $\mu = 0.5$	$40\pi\mu I^-_{\beta=0,1}(0,\mu)/t_{01}(1)$, $\mu = 1.0$	$10[1.0 - q_{z\beta=0,1}(0)]$
Direct[4]	2.001094	5.44812	8.74655	0.92609
Ambar[4]	2.001031	5.44790	8.74616	0.92604
Current 1L	1.999625	5.44448	8.74038	0.92544
Current 2L	2.00037	5.44628	8.74066	0.92577

$n_{10} = 1.50$, $n_{12} = 1.00$, $\tau_o = 8.0$, $\omega_1 = 0.7$, $\beta = 0.0$.

Table 3 Comparison of current results with those of Dougherty[14]

	$S_{\beta=0,1}(0)/t_{01}(1)$	$S_{\beta=0,1}(\tau_o)/t_{01}(1)$	$I^-_{\beta=0,1}(0,1) \times 4\pi\mu/t_{01}(1)$	$T(\mu = 1)$
Direct[14]	3.857896	2.557394	2.101162	1.965850
Current 1L	3.80523	2.556909	2.100681	1.965385

$n_{10} = 1.50$, $n_{12} = 1.20$, $\tau_o = 1.0$, $\omega_1 = 1.0$, $\beta = 0.0$.

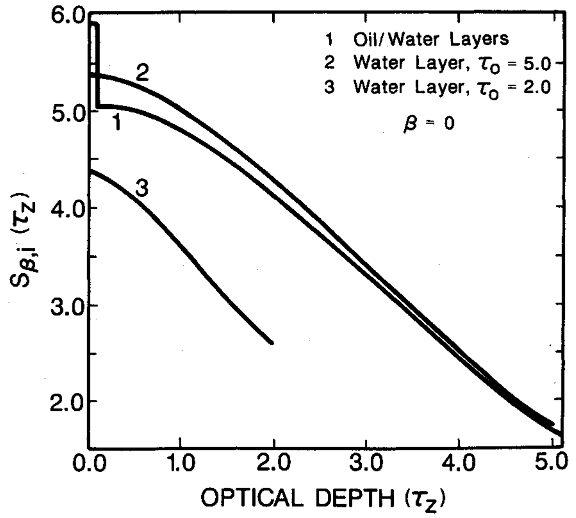

Fig. 3 One-dimensional source function inside the medium (for oil: $\omega = 0.9$ and $n = 1.48$; for water: $\omega = 1.0$ and $n = 1.33$).

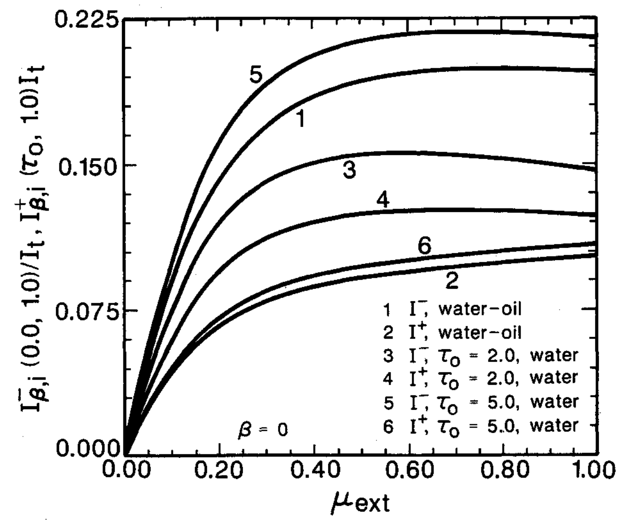

Fig. 4 The intensity outside a one-dimensional medium: one- and two-layer effects (for water only: $\omega = 1.0$; for water–oil: $\omega = 1.0$ and $\tau_o = 5.0$ for water, while $\omega = 0.9$ and $\tau_o = 0.1$ for oil).

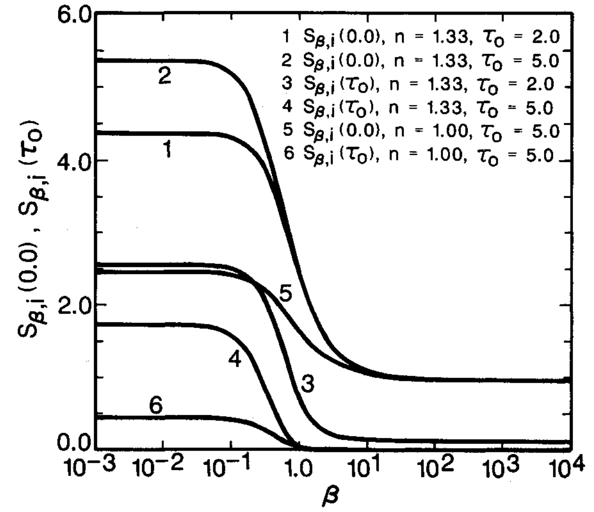

Fig. 5 Effect of refractive index and optical thickness on the β-varying source function for $\omega = 1.0$. (Curves 5 and 6 are from Ref. 16).

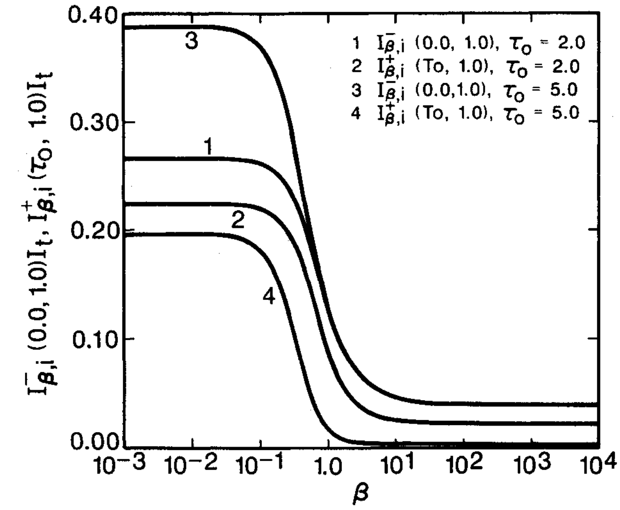

Fig. 6 Effect of optical thickness on the β-varying intensity for $\omega = 1.0$ and $n = 1.33$.

dramatic drop in the source function just at the interface between the two layers. This drop is due to the change in index of refraction from 1.48 to 1.33, which results in more radiation being reflected back into the thin layer. The thin layer exerts less influence on the source function in the medium as we move away from the interface. This is shown by comparing curve 2—the source function for the same layer of water ($\tau_o = 5.0$) but without the second thin layer—to the curve resulting from the addition of the thin layer. The third curve ($\tau_o = 2.0$, water only) on Fig. 3 is included to show the effects of varying the optical thickness of the layer on the source function. Small optical thickness means less scattering, and hence a smaller source function.

Another special case of interest is the optical thickness and the index of refraction effects on the angular distribution of the back-scattered intensity (I^-) and the transmitted intensity (I^+). The one-dimensional solution, which is still the special case of $\beta = 0$, gives satisfactory insight into these effects. Figure 4 gives the back-scattered intensity (I^-) and the transmitted intensity (I^+) outside the medium for the cases of one- and two-layer media. For the one-layer medium ($n = 1.33$, $\omega = 1.0$), comparing $\tau_o = 2$ and $\tau_o = 5$, the back-scattered intensity is increased for the larger τ_o. This trend is reversed for the transmitted intensity. When a thin layer ($\tau_o = 0.1$ and $\omega = 0.9$) of paraffin oil is introduced on top of the water layer (for the case where $\tau_o = 5$), both transmitted and back-scattered intensities are reduced drastically.

Two-Dimensional Results

Before presenting some two-dimensional results, we will examine the β-varying solution. Figure 5 presents the influence of optical thickness τ_o, spatial frequency β, and index of refraction on the source function. The effects of these parameters on back-scattered intensity and transmitted intensity are illustrated in Figs. 6 and 7. The source function at the top and bottom of the medium exhibits similar trends for both optical thicknesses. For small β values, the source function at the top is larger for the smaller τ_o. This trend reverses for the bottom boundary source function. These functions are compared to the results by Crosbie and Koewing[16] for unit refractive index. It is found from the comparison that the index of refraction increases the β-varying results for the source function, but reduces the intensity results (Figs. 5 and 7). Note that, regardless of optical thickness and refractive index, the β-varying source function results approach nearly the same asymptote as β increases. As β becomes very large, the results represent small τ_r behavior (see Refs. 5 and 6).

After obtaining the β-varying solution, numerical integration [e.g., Eqs. (31)] gives the τ_r-varying solution. Figures 8

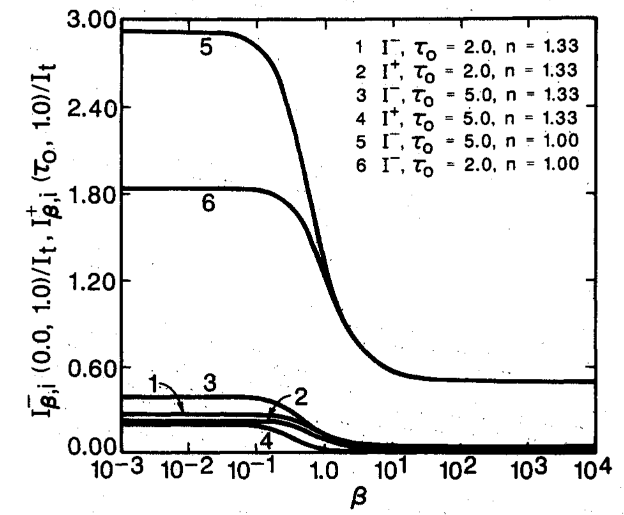

Fig. 7 Effects of refractive index and optical thickness on the β-varying intensity for ω = 1.0. (Curves 5 and 6 are from Ref. 16.)

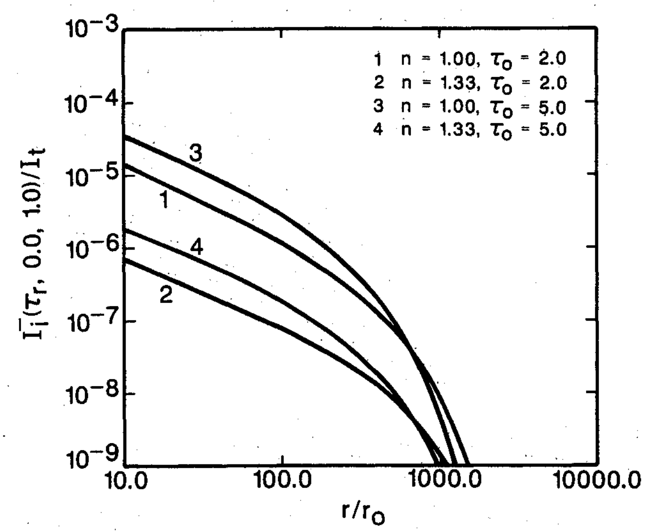

Fig. 8 Effect of refractive index and optical thickness on reflected intensity far from the incident beam for ω = 1.0. (β-Varying results of Ref. 16 were used to obtain curves 1 and 3.)

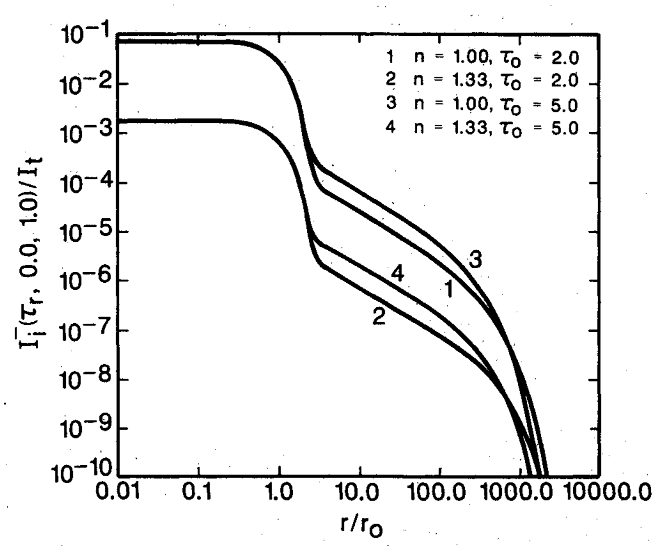

Fig. 9 Effect of refractive index and optical thickness on reflected intensity for ω = 1.0. (β-Varying results of Ref. 16 were used to obtain curves 1 and 3.)

and 9 are plots of the reflected intensity outside the medium. The first plot (Fig. 8) provides a close look at the trends over the normal experimental measurement range.[15] Figure 9 gives the intensity over a wider range, specifically within the original incident laser beam. The trends clearly show a dramatic increase in the reflected intensity within the original laser beam—which is to be expected. Curves 2 and 4 are for the reflected intensity from a one-layer medium with top and bottom reflecting boundaries (n = 1.33). These curves show that the back reflected intensity increases in value as the optical thickness of the medium increases. This is due to the increased amount of scattering [events]. Some results for unit refractive index[16] are also included in Fig. 9 for comparison. It is apparent from the plots that assuming unit refractive index in the case of a water medium would lead to severe overestimation of the reflected intensity. However, the general trends of the two sets of solutions are quite similar.

Conclusions

The theoretical work presented herein will provide a more flexible and general tool to study a wider range in indices of refraction, and to investigate their effect on the back-scattered intensity, than is available in the existing models. The computerized model has the capability of varying the refractive index, along with handling internal reflection at the boundaries and allowing the albedo to vary through layers of variable optical thickness of the medium being analyzed. With such a model, validated by experimental work (now being conducted at Oklahoma State University[15]), radiative transfer within simulated solar ponds or any other multilayered medium can be studied for a much wider variety of parameters than those assessed by the hardware.

Appendix: Auxiliary Functions

In this appendix, several functions that were referenced in the body of this paper are written explicitly. In the following equations, N represents the number of layers, and i represents a particular layer:

$$S^c_{io}(\mu_i,\tau_z) = t_{01}(\mu_o)D_N^{-1}(\mu_i)A_i(\mu_i)\xi_{io}(\mu_i)\Phi^1_{i1}(\mu_i)\psi^1_i(\mu_i,\tau_z) \quad \text{(A1)}$$

$$t_{ij}(\mu_i) = [1 - \rho_{ij}(\mu_i)] \quad \text{(A2)}$$

$$D_N(\mu_i) = 1 - \sum_{k=2}^{N} Q_{1k}(\mu_i) \quad \text{(A3)}$$

$$A_i(\mu_i) = [1 - \rho_{i,i-1}(\mu_i)\rho_{i,i+1}(\mu_i)\exp(-2\Delta\tau_i/\mu_i)]^{-1} \quad \text{(A4)}$$

ξ is defined in the following equations:

$$\xi_{ij}(\mu_i) = \prod_{k=v}^{w}[\exp[-\Delta\tau_k/\mu_k]], \quad \text{for} \quad \text{abs}(j - i) > 1 \quad \text{(A5)}$$

where v = minimum (i,j) + 1, and w = maximum (i,j) − 1, and $\xi_{ij}(\mu_i) = 1$, for abs$(j - 1) \leq 1$. The Φ's are defined as follows:

$$\Phi^1_{ij}(\mu_i) = \prod_{k=j}^{i-1} [A_k[\mu_k]\, t_{k,k+1}[\mu_k]], \; i > j \quad \text{(A6)}$$

where $\mu_k = [1 - (1 - \mu_i^2)n_{ik}^2]^{1/2}$. If $i = j$, then $\Phi^1_{ij}(\mu_i) = 1$

$$\Phi^2_{ij}(\mu_i) = \prod_{k=i+1}^{j} [A_k[\mu_k]\, t_{k,k-1}[\mu_k]], \; i < j \quad \text{(A7)}$$

If $i = j$, the $\Phi^2_{ij}(\mu_i) = 1$. The definitions for ψ are

$$\psi^1_i(\mu_i,\tau_z) = T^1_i(\mu_i)\exp[(\tau_{zi-1} - \tau_z)/\mu_i] + R^1_i(\mu_i)\exp[(\tau_z - \tau_{zi} - \Delta\tau_i)/\mu_i] \quad \text{(A8)}$$

$$\psi_i^2(\mu_i,\tau_z) = T_i^2(\mu_i)\exp[(\tau_z - \tau_{zi})/\mu_i] + R_i^2(\mu_i)\exp[(\tau_{zi-1} - \tau_z - \Delta\tau_i)/\mu_i] \quad \text{(A9)}$$

Z is defined as

$$Z_i^+(\mu_i) = t_{01}(\mu_o)T_i^1(\mu_i)\xi_{io}(\mu_i)\Phi_{i1}^1(\mu_i) \quad \text{(A10)}$$

$$Z_i^-(\mu_i) = t_{01}(\mu_o)R_i^1(\mu_i)\xi_{io}(\mu_i)\Phi_{i1}^1(\mu_i)\exp(-\Delta\tau_i/\mu_i) \quad \text{(A11)}$$

R_i and T_i are given by

$$R_i^1(\mu_i) = \rho_{i,i+1}(\mu_i) + (1 - \delta_{iN})\sum_{k=i+1}^{N} K_{ik}(\mu_i) \quad \text{(A12)}$$

$$R_i^2(\mu_i) = \rho_{i,i-1}(\mu_i) + (1 - \delta_{i1})\sum_{k=1}^{i-1} K_{ik}(\mu_i) \quad \text{(A13)}$$

$$T_i^1(\mu_i) = 1 - \sum_{k=i+2}^{N} Q_{i+1,k}(\mu_i), \quad \text{for} \quad (i+2) \le N \quad \text{(A14)}$$

If $i > N - 2$, then $T_i^1(\mu_i) = 1$

$$T_i^2(\mu_i) = 1 - \sum_{k=2}^{i-1} Q_{1k}(\mu_i), \quad \text{for} \quad i > 2 \quad \text{(A15)}$$

If $i \le 2$, then $T_i^2(\mu_i) = 1$. For Γ, when $i < j$

$$\Gamma_{ij}(\mu_i',\tau_z,\tau_z') = n_{ij}^2\Phi_{ij}^2(\mu_i')\xi_{ij}(\mu_i')\psi_j^1(\mu_i',\tau_z')\psi_i^2(\mu_i',\tau_z) \quad \text{(A16)}$$

When $i > j$

$$\Gamma_{ij}(\mu_i',\tau_z,\tau_z') = n_{ij}^2\Phi_{ij}^1(\mu_i')\xi_{ij}(\mu_i')\psi_j^2(\mu_i',\tau_z')\psi_i^1(\mu_i',\tau_z) \quad \text{(A17)}$$

and when $i = j$, then

$$\Gamma_{ii}(\mu_i',\tau_z,\tau_z') = R_i^2(\mu_i')\psi_i^1(\mu_i',\tau_z') \times \exp[-(\tau_{zi-1} - \tau_z)/\mu_i'] + R_i^1(\mu_i')\psi_i^2(\mu_i',\tau_z') \times \exp[-(\tau_{zi} - \tau_z)/\mu_i'] \quad \text{(A18)}$$

For the $\boldsymbol{B}$ values

$$\boldsymbol{B}_{ij}^{1,2}(\mu_i,\tau_r) = n_{ij}^2\Phi_{ij}^{2,1}(\mu_i)\xi_{ij}(\mu_i) \times \int_{\tau_{zj-1}}^{\tau_{zj}} S_j(\tau_r',\tau_z')\Psi_j^{1,2}(\mu_j,\tau_z')\,d\tau_z'/\mu_j \quad \text{(A19)}$$

$$\boldsymbol{B}_{\beta,ij}^{1,2}(\mu_i) = n_{ij}^2\Phi_{ij}^{2,1}(\mu_i)\xi_{ij}(\mu_i) \times \int_{\tau_{zj-1}}^{\tau_{zj}} S_{\beta,j}(\tau_z')\Psi_j^{1,2}(\mu_j,\tau_z')\,d\tau_z'/\mu_j \quad \text{(A20)}$$

The two functions Q and K are given only for four layers. For abs$(j - 1) = 1$, Q and K are given as

$$Q_{nj}(\mu_i) = G_{n,n-1}(\mu_n)G_{j,j+1}(\mu_j) \quad \text{(A21)}$$

$$K_{ij}(\mu_i) = t_{ij}(\mu_i)G_{j,2j-i}(\mu_j) \quad \text{(A22)}$$

For abs$(j - i) = 2$, Q and K are given as

$$Q_{nj}(\mu_i) = A_k(\mu_k)\exp(-2\Delta\tau_k/\mu_k)t_{kj}(\mu_k)G_{jm}(\mu_j) \times [\rho_{kn}(\mu_k) + t_{kn}(\mu_k)G_{nl}(\mu_n)] \quad \text{(A23)}$$

$$K_{ij}(\mu_i) = A_k(\mu_k)\exp(-2\Delta\tau_k/\mu_k)t_{kj}(\mu_k)G_{jm}(\mu_j) \times [t_{ik}(\mu_i)t_{ki}(\mu_k) - \rho_{ik}(\mu_i)\rho_{ki}(\mu_k)] \quad \text{(A24)}$$

where $k = (i + j)/2$, $m = (3j - i)/2$, $l = (3i - j)/2$. And for abs$(j - i) = 3$, Q and K are given as

$$Q_{nj}(\mu_i) = A_k(\mu_k)\exp(-2\Delta\tau_k/\mu_k) \times t_{kj}(\mu_k)G_{jm}(\mu_j) \times \{t_{lk}(\mu_l)t_{kl}(\mu_k)A_l(\mu_l) \times \exp(-2\Delta\tau_l/\mu_l)[\rho_{ln}(\mu_l) + t_{ln}(\mu_l) \times G_{nr}(\mu_n)] + \rho_{kl}(\mu_k)[1 - G_{nr}(\mu_n)G_{lk}(\mu_l)]\} \quad \text{(A25)}$$

$$K_{ij}(\mu_i) = A_k(\mu_k)\exp(-2\Delta\tau_k/\mu_k)t_{kj}(\mu_k)G_{jm}(\mu_j) \times\{t_{il}(\mu_i)t_{li}(\mu_l)A_l(\mu_l)\exp(-2\Delta\tau_l/\mu_l) \times [t_{lk}(\mu_l)t_{kl}(\mu_k) - \rho_{lk}(\mu_l)\rho_{kl}(\mu_k)] - \rho_{il}(\mu_i)[\rho_{kl}(\mu_k) + t_{kl}(\mu_k)G_{li}(\mu_l)]\} \quad \text{(A26)}$$

where $k = (2j + i)/3$, $l = (j + 2i)/3$, $m = (4j - i)/3$, and $r = (4l - j)/3$. For all of the above definitions of Q_{nj} and K_{ij}, the μ values are related as follows: $\mu_\varepsilon = [1 - (1 - \mu_i^2)n_{i\varepsilon}^2]^{1/2}$, where $\varepsilon = j, k, l, m, n$, or r.

In addition, the G_{ij} values are

$$G_{ij}(\mu_i) = A_i(\mu_i)\rho_{ij}(\mu_i)t_{i,2i-j}(\mu_i)\exp(-2\Delta\tau_i/\mu_i) \quad \text{(A27)}$$

if $\mu_i < \mu_{i,2i-j,\mathrm{cr}}$, then $G_{ij}(\mu_i) = 0$.

Lastly, for cases where the number of layers is greater than one, the equation for τ_r' [Eq. (4)] should be modified to include passage through all intermediate layers as follows:

$$(\tau_r')^2 = (\tau_r)^2 + (\tau_{\alpha ij})^2 - 2\tau_r(\tau_{\alpha ij})\cos(\phi') \quad \text{(A28)}$$

where

$$\tau_{\alpha ij} = \sum_{k=\min(i,j)}^{\max(i,j)} |\tau_{zk-1} - \tau_{zk}|(1 - \mu_k'^2)^{1/2}/\mu_k' \quad \text{(A29)}$$

and $\mu_k' = [1 - (1 - \mu_i'^2)n_{ik}^2]^{1/2}$. Note that when $k = \min(i,j)$, $\tau_{zk-1} = \tau_z$, and when $k = \max(i,j)$, $\tau_{zk} = \tau_z'$. In the special case where $i = j$, Eq. (A29) becomes Eq. (5).

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